

MEASURE ESTIMATES, HARNACK INEQUALITIES AND RICCI LOWER BOUND

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ABSTRACT. Consider a Riemannian metric-measure space. We establish an Alexandrov-Bakelman-Pucci type estimate connecting the Bakry-Émery Ricci curvature lower bound, the modified Laplacian and the measure of certain special sets. We apply this estimate to prove Harnack inequalities for the modified Laplacian operator (and fully non-linear operators, see the Appendix). These inequalities seems not available in the literature; And our proof, solely based on the ABP estimate, does not involve any Sobolev inequalities nor gradient estimates. We also propose a question regarding the characterization of Ricci lower bound via the Harnack inequality.

1. INTRODUCTION

This paper is devoted to generalize Alexandrov-Bakelman-Pucci (abbrev. ABP) techniques to general Riemannian setting and use them to study the relation between Ricci lower bound and elliptic PDEs on Riemannian metric-measure spaces. In particular, we establish an ABP-type inequality (Thm.1.2), which connects the measure of some specific sets (contact sets, Defn.1.1) and N -Bakry-Émery (abbrev. BE) Ricci curvature ($N \in [n, \infty]$). The idea of this work is largely influenced by the remarkable paper of Cabré ([2]).

To illustrate the power of the ABP-techniques, we shall consider, on a smooth Riemannian metric-measure space (\mathcal{M}, g, ν) with $\nu = e^{-V} \text{vol}_g$, the modified Laplacian operator

$$\Delta_\nu u = \Delta u - g(\nabla u, \nabla V) \Leftrightarrow L_{\nabla u} \nu = (\Delta_\nu u) \nu,$$

(L stands for the Lie derivative). We shall prove the Harnack inequalities (Thm.1.6–Thm.1.8) for this operator under assumption of local lower bound of N -BE Ricci curvature ($N < \infty$). Harnack inequalities in this generality seem to be unavailable in the literature; and our proof differs completely from standard methods in geometric analysis. In particular, it does not involve Sobolev inequalities nor gradient estimates. This proof also applies for fully-nonlinear operators (see the Appendix) and it suggests us to consider characterizing Ricci lower bound by the Harnack inequality.

ABP-techniques are of central importance in the modern study of the elliptic equations. They have been widely applied in the study of various classes of linear and non-linear elliptic equations in the Euclidean space (see [3] and reference therein). However,

they seem not much recognized in the field of geometric analysis and Riemannian geometry. Part of the reasons that limit the application of these techniques in Riemannian geometry are the following.

Fix a given function u (solution to some PDE), a key idea in ABP techniques is to consider the set of minimum points of $u - l$ for each possible linear function l , and consider its image under the gradient mapping ∇u . While non-constant linear functions do not in general exist on a Riemannian manifold; and ∇u , whose image lies in the tangent bundle, seems difficult to deal with.

These difficulties are resolved by Cabré in [2]. In this pioneer work, Cabré proposed to replace linear functions by paraboloids–squared distance functions $\rho^2(\cdot, y)$ – and consider the following special sets:

Definition 1.1. Let Ω be a bounded subdomain of the smooth Riemannian manifold (\mathcal{M}, g) and $u \in C(\Omega)$. For a given $a \geq 0$ and E compact subset of Ω , the contact set of opening a is defined by

$$A(a, E/\Omega, u) := \{x \in \overline{\Omega}, \inf_{\overline{\Omega}} \{u + a\rho^2(\cdot, y)\} = u(x) + a\rho^2(x, y) \text{ for some } y \in E\},$$

where ρ is the distance function of the metric g .

And replace the gradient map ∇u by the map

$$(1.1) \quad F[u](x) := \exp_x(\nabla u(x)), \quad u \in C^2(M).$$

Based on this idea, Cabré was able to control the integral of the Laplacian Δu (or more generally non-divergence linear operator, see Defn.1.1 in [2]) over a sub-level set from below by the volume of the domain (see Lemma 4.1 in [2]). Then following the approach of Krylov-Safanov ([12], [13]), Harnack inequalities (for non-divergence equation) on spaces with non-negative sectional curvature are derived from this estimate via Calderón-Zygmund decompositions. Recently, the approach in [2] has been extended by S. Kim ([11]), who replaced the assumption of non-negative sectional curvature by certain balanced condition on sectional curvatures according to the given operator L (see condition 4 and 5 in Sec.1 of [11]). In particular, for the case of Laplacian, that condition is equivalent to Ricci nonnegative.

Nevertheless, power of Cabré’s approach has not yet been fully explored. Following his approach, combined with some recent development in the theory of optimal transport ([25] and reference therein), we can extend the ABP techniques with a considerably larger generality – only local BE-Ricci bound required (see. Defn.2.1 for the BE-Ricci curvatures). In particular, we prove the following *Measure Estimate Formula*, which resembles the Euclidean version of ABP estimate.

Theorem 1.2. Let (\mathcal{M}, g, ν) be a complete Riemannian metric-measure space with dimension $n \geq 2$. Let E be a closed subset of a geodesic ball B_r and $u \in C(\overline{B_r})$; Let $K \geq 0$ and $N \in [n, \infty]$ be two constants.

Suppose $a > 0$ and $A(a, E/B_r, u) \subset B_r$; Suppose there exists a subdomain Ω' containing $A(a, E/B_r, u)$ such that $u \in C^2(\Omega')$. Then the following statements holds:

If $N < \infty$, denote $\omega_{K,N} = 2\sqrt{K/N}$, then

$$\text{Ric}_{N,\nu}|_{B_r} \geq -Kg \Rightarrow \nu[E] \leq \int_{A(a,E/B_r,u)} \left\{ \mathcal{D}_{K,N,r}[u/a](x) \right\}^N \nu(dx)$$

where for any $x \in A(a, E/\Omega, u)$,

$$\mathcal{D}_{K,N,r}[u/a](x) := \frac{\sinh(r\omega_{K,N})}{r\omega_{K,N}} \left[r\omega_{K,N} \coth(r\omega_{K,N}) + \frac{\Delta_\nu u(x)}{Na} \right]$$

In the case $K = 0$, expressions are understood as their obvious limits.

If $N = \infty$

$$\text{Ric}_{\infty,\nu}|_{B_r} \geq -Kg \Rightarrow \nu[E] \leq \int_{A(a,E/B_r,u)} \exp \left\{ \mathcal{D}_{K,N,r}[u/a](x) \right\} \nu(dx)$$

where

$$\mathcal{D}_{K,N,r}[u/a](x) := 2r^2 K + \frac{\Delta_\nu u(x)}{a}.$$

Remark 1.3. The upshot of the above formula is that the integration is done only on a special set—the contact set, and the lower bound of this integral can be controlled. This is the essence of the Euclidean ABP estimate. Unlike the ABP estimate in Euclidean space, the above formula does not involve infima of the unknown function. However, we shall see this would not limit its application.

Remark 1.4. We have assumed the underlying manifold is smooth and u is C^2 near $A(a, E/\Omega, u)$. In fact, we only need u to be semi-concave (see, Defn.16.4, p.429 in [25]) near A . Moreover the theorem can be established on Alexandrov spaces. However, to avoid heavy formulations and to better present the main ideas, we shall stay with C^2 -functions. Our proof can be easily adopted to including semi-concave functions and Alexandrov spaces.

Remark 1.5. Refer to the proof of Thm.12.4 in the Appendix for generality and sufficiency of only considering $\Delta_\nu u$ in the above formula. Indeed, on the contact set other linear (appropriate nonlinear) operators can be controlled from below by the Laplacian.

We want to remark that the Measure Estimate formula is valid in all effective dimension, including particularly the case $N = \infty$. We believe this formula will have applications in many geometric problems. We shall discuss some of these applications in a separate paper.

The underline idea of proving the above theorem is indeed contained in [2]. That is, apply the Area formula to the map $F[u]$ (Eq.1.1) on contact set A . Here, rather than use the direct calculation of Jacobi determinant of $F[u]$ given in [2], we employ an ODE comparison estimate suggested in Ch.14 of [25]. Besides allowing us to establish the estimate for very general curvature condition, this ODE estimate matches in a remarkable way the fine structures of contact sets (see. Lem.3.8, Lem.4.6).

Similar to that in [2] (and [11]), the Krylov-Safanov method allows one to deduce the following Harnack inequalities from Thm.1.2.

Given $K \geq 0, N \in [n, \infty)$, denote

$$(1.2) \quad \eta = \eta_{K,N,2R} = 1 + 8R \log 2 \sqrt{K/N}$$

in the following statements. All integrals are preformed against ν . The manifold \mathcal{M} has dimension $n \geq 2$.

Theorem 1.6. *Let (\mathcal{M}, g, ν) be a complete smooth Riemannian metric-measure space. Let $K \geq 0$ and $N \in [n, \infty)$. Let $u \in C^2(B_{2R}) \cap C(\overline{B}_{2R})$ and $f \in C(B_{2R})$. Suppose*

$$\text{Ric}_{N,\nu}|_{B_R} \geq -Kg \quad \Delta_\nu[u] \leq f \text{ in } B_{2R}, \quad u \geq 0 \text{ in } B_{2R}$$

Then,

$$(1.3) \quad \left(\int_{B_{R/2}} u^{p_0} \right)^{1/p_0} \leq C_0 \left\{ \inf_{B_{R/2}} u + R^2 \left(\int_{B_{2R}} |f|^{N\eta} \right)^{1/(N\eta)} \right\}$$

where p_0, C_0 are constants only depending on $\sqrt{K}R, N$. Moreover $C_0 = e^{2/p_0}$.

Theorem 1.7. *Let (\mathcal{M}, g, ν) be a complete smooth Riemannian metric-measure space. Let $K \geq 0$ and $N \in [n, \infty)$. Let $u \in C^2(B_{2R}) \cap C(\overline{B}_{2R})$ and $f \in C(B_{2R})$*

Suppose

$$\text{Ric}_{N,\nu}|_{B_R} \geq -Kg \quad \Delta_\nu[u] \geq f \text{ in } B_{2R}.$$

Then, for any $p > 0$

$$(1.4) \quad \sup_{B_{R/2}} u \leq C_2(p) \left\{ \left(\int_{B_R} (u^+)^p \right)^{1/p} + R^2 \left(\int_{B_{2R}} |f|^{N\eta} \right)^{1/(N\eta)} \right\}$$

where $C_1(p)$ is a constant only depending on $\sqrt{K}R, N$ and p .

Theorem 1.8. *Let (\mathcal{M}, g, ν) be a Complete Smooth Riemannian Metric-Measure Space. Let $K \geq 0$ and $N \in [n, \infty)$. Let $u \in C^2(B_{2R}) \cap C(\overline{B}_{2R})$ and $f \in C(B_{2R})$*

Suppose

$$\text{Ric}_{N,\nu}|_{B_R} \geq -Kg \quad \Delta_\nu[u] = f \text{ in } B_{2R}, \quad u \geq 0 \text{ in } B_{2R}$$

Then

$$(1.5) \quad \sup_{B_{R/2}} u \leq C_2 \left\{ \inf_{B_{R/2}} u + R^2 \left(\int_{B_{2R}} |f|^{N\eta} \right)^{1/(N\eta)} \right\}$$

where C_2 is a constant only depending on $\sqrt{K}R$ and N .

Remark 1.9. In the case $K = 0$, the integral expressions of the right-hand side f reduce to the standard averaged L^N -norm. This agree with the Harnack inequalities in [2] when $K = 0$ and $\nu = \text{vol}_g$. Increasing the exponent of integration by a factor depending on Ricci curvature lower bound is necessary. This can be seen easily from the examples in (K, N) -Hyperbolic space.

Remark 1.10. For the readers' convenience, a set of explicit estimate of $p_0, C_1(p_0), C_2$ shall be given at the end of §2 (see Lem.2.9). For general p , $C_1(p)$ is obtained from $C_1(p_0)$ by interpolation (see. the proof in §10).

Remark 1.11. We emphasize that constants (p_0, C_1, C_2, η) in above theorems depend on the product \sqrt{KR} , rather on R or K alone. In particular, if $K = 0$, p_0, C_1 , and C_2 are independent of radius R and $\eta = 1$. Hence the above theorems recover the Harnack inequalities (for the Laplacian) in [2] and [11] (see the Appendix for case of fully-nonlinear uniform elliptic equations).

Remark 1.12. Higher regularity estimates, including gradient and C^2 estimates, can be obtained from the Harnack inequalities via the methods given in [3].

Besides proving Harnack inequalities in larger generality, we provide a different presentation of Krylov-Safanov argument in proving Harnack inequalities. In this presentation, the Calderón-Zygmund decomposition (used in [2]) is replaced by Vitali's covering lemma. Though it follows essentially same spirits as that in [2], the argument here seems more elementary and transparent. It is similar to the covering argument used by L.A. Caffarelli in his breaking through work on real Monge-Ampère equations ([4]). We learned this argument from O. Savin's lecture at Columbia University and his paper [23].

Harnack inequalities has been intensively studied in geometric analysis. An incomplete list includes: the remarkable work of S.Y. Cheng and S.-T Yau [8], who considered $\Delta u = -\lambda u$, $\lambda > 0$ and proved Harnack inequality for solutions via establishing a sharp gradient estimate; Later, J. Cheeger, M. Gromov, M. Taylor, S.Y. Cheng, P. Li and S.-T. Yau has employed the method of De Giorgi-Nash-Moser iteration and C.B. Croke's work on Sobolev inequalities ([9]) to consider differential inequalities ([6], [7]). The optimal results via this approach is given by L. Saloff-Coste ([22]). Based on the penetrating work on Sobolev inequalities due to N. Varopoulos ([24]) and Cheng and Yau's gradient estimate, he proved the Harnack inequality for divergent operators on manifolds with standard Ricci curvature bounded from below. These works have also been extended to various general cases. For example, in [28], [26], these authors studied the gradient estimate for p -harmonic function on Riemannian manifolds. Recently, Li [14] (see also [1]) followed the main line of [8] and gave the Harnack inequality for *solutions* of the modified Laplacian equation on Riemannian Metric-Measure space.

Comparing to the above mentioned work, besides the generality of our results, we would like to emphasize that our approach, following Cabré, differs completely from above mentioned work.

The statement of the Harnack inequality (Thm.1.8) and the key ingredient (Thm.1.2) in our proof suggests us that constants in Harnack inequalities are of geometric meaning. In fact, we believe that these constants characterize the lower bound of Ricci curvature (for Riemannian metric-measure space, also related to the effective dimension). Based on this point, we propose some questions concerning the relation between constants in the Harnack inequality and the lower bound of Ricci curvature. In the present paper, we shall provide a precise formulation of these questions in this paper and suggest some affirmative evidence. We think this kind of characterization would have applications in

the study of Gromov-Hausdorff convergence, geometric flows and Alexandrov spaces. Our idea here is largely inspired by the work of J. Lott, C. Villani ([17], [16]).

The paper is organized as follows: In §2, we fix our notations and conventions. In particular, we give a full list of constants involved in the later proof. §3 and §4 are devoted to study the contact sets and the Jacobi determinant of $dF[u]$ respectively. In these two section, we shall see how the contact sets, the Jacobi fields and the underlying geometry interact with each other. In §5, we prove the Measure Estimate Formula – Thm.1.2. §6 and §7 contains some preparations for the proof of Harnack inequalities (Thm.1.6–Thm.1.8). §8 contains the main technical lemma in proving these theorems. In §9 and §10, Thm.1.6–Thm.1.8 are proved. In §11, we discuss a possible way to characterize the Ricci lower bound by Harnack inequalities. In the Appendix, we extend the method in this paper to prove the Harnack inequalities for fully-nonlinear uniform elliptic operators! on Riemannian manifolds.

2. NOTATIONS, CONVENTIONS AND CONSTANTS

In order to avoid any potential confusion, we first state our convention regarding the curvatures and the cut-locus.

- *Riemannian Metric-Measure space*: In the paper, the background manifold is the Riemannian metric-measure space (\mathcal{M}, g, ν) where g is the Riemannian metric on \mathcal{M} and $\nu = e^{-V} \text{vol}$ is a reference measure with $V : M \rightarrow \mathbb{R}$ a C^2 function. Notice that, if $V = 0$, then ν is just the usual volume measure vol_g .

- *Curvatures*: Recall the definition of Riemannian curvature tensor

$$(2.1) \quad \text{Riem}(X, Y) := D_Y D_X - D_X D_Y + D_{[X, Y]}, \quad X, Y \in T\mathcal{M}$$

and that of Ricci curvature

$$(2.2) \quad \text{Ric}_x(Z) := \sum_i g_x(\text{Riem}(Z, e_i)Z, e_i), \quad Z \in T_x\mathcal{M}$$

where e_i is the orthogonal basis w.r.t. g . Note here our convention on Ric is standard, and it agrees with both the reference [25] and [20]; our convention of Riem agrees with the reference [25] (p.371) but differs from [20](p.33) a sign.

We also recall the following definition:

Definition 2.1. The Bakry-Emery (abbrev. BE) Ricci curvature associate to ν with effective dimension $N \in [n, \infty]$ is defined by

$$\text{Ric}_{N, \nu} = \begin{cases} \text{Ric} & N = n, \\ \text{Ric} + D^2V - \frac{DV \otimes DV}{N-n} & N > n \\ \text{Ric} + D^2V & N = \infty. \end{cases}$$

Here, we assume $V = 0$ whenever $N = n$.

Remark 2.2. For the importance and the geometry of the Bakry-Emery Ricci curvature, one may refer to [15], [25], the recent work of [27] and the reference therein. In particular, $\text{Ric}_{\infty, \nu}$ plays an important role in Perelman's work ([19]) on Hamilton's Ricci flow. .

- *Cut-locus:* We recall the definition of cut-points and focal points. We follow the convention in [25] (p.193). Note, this convention may differ from some text, but will not affect the generality of this paper.

Definition 2.3. Fix $x \in (\mathcal{M}, g)$, a point y is called a cut point of x if there is a geodesic $\gamma(t)$ such that $\gamma(0) = x$ and $\gamma(t_c) = y$ and satisfies that i) $\gamma(t)$ is minimizing for all $t \in [0, t_c)$ and ii) $\gamma(t_c + \epsilon)$ is not minimizing for any $\epsilon > 0$.

Definition 2.4. Two points x and y are said to be focal (or conjugate) if y can be written as $\exp_x(tW)$, $W \in T_x \mathcal{M}$, and the differential $d|_W \exp_x(\cdot)$ is not invertible.

Definition 2.5. Given a point $x \in \mathcal{M}$, the cut-locus $\text{Cut}(x)$ of x is the set consisting of all cut-points and focal (conjugate) points of x .

Remark 2.6. Being cut-points and focal points are symmetric relations. $x \in \text{Cut}(y)$ if and only if $y \in \text{Cut}(x)$.

- *Contact Relations:* We recall the following terminologies:

Definition 2.7. Let Ω be a subdomain of \mathcal{M} . Let u and φ be two continuous functions in Ω .

Let $x_0 \in \Omega$ and U be a subset (not necessarily open) of Ω , we say φ touches u from above (resp. below) at x_0 in U if $\varphi(x) \geq w(x)$ (resp. $\varphi(x) \leq w(x)$) for all $x \in U$ and $\varphi(x_0) = w(x_0)$.

We say φ touches u from above (resp. below) at x_0 if there is a neighborhood U of x_0 such that φ touches u from above (resp. below) at x_0 in U .

- *Convention in Notations* We also have the following conventions in notations.

i) Throughout this paper, a later C , *Without any Subscript*, represents a *Pure* constants *Greater* than 1. It might change from line to line. However we emphases that it does *Not* depends on any parameter. Moreover, to make the proof more transparent, we shall try to minimize the usage of C and try to be explicit.

ii) We always denote the standard dimension of \mathcal{M} by n , and we assume throughout the paper that $n \geq 2$.

iii) We always denote $B_r(x)$ to be the geodesic ball of radius r centered at x . We shall omit the center x when it has no particular importance and also cause no confusion.

iv) Throughout the paper, integrals are performed against the measure ν ; and the distance function is denoted by ρ . The notation ρ_y means the distance from a fixed point y .

• *Special Functions and Notations* In the rest of this paper, in particular the proof of our main theorems, many parameters and functions get involved. In order to give a clear presentation, we shall make several short-hand notions. Here, we list these notations and some basic facts regarding them.

i) Let $K \geq 0, N \in [n, \infty)$ and $r > 0$, we denote

$$(2.3) \quad \omega_{K,N} := 2\sqrt{K/N}, \quad \mathcal{D}_{K,N,r} := 2^N e^{4r\sqrt{NK}},$$

and define also

$$(2.4) \quad \eta_{K,N,r} := \log \mathcal{D}_{K,N,r} / (N \log 2) = 1 + 4r\sqrt{NK} \log 2$$

Notice: The subscript K, N, r does *Not* mean $\mathcal{D}_{N,K,r}$ or $\eta_{K,N,r}$ depends on K nor r in a separated way. It is just for convenience. When no confusion arise, one or all subscripts might be omitted. In particular, we shall often use \mathcal{D}_r, η_r in replacing $\mathcal{D}_{K,N,r}$ and $\eta_{K,N,r}$. However, this is only for convenience. Again, it does not mean η depends on r nor K in a separated way.

ii) Let $t \in [0, \infty)$, define

$$(2.5) \quad \mathcal{H}(t) := t \coth(t), \quad \mathcal{S}(t) := \sinh(t)/t$$

Note \mathcal{H}, \mathcal{S} are differentiable and have positive derivative for all $t > 0$; $\mathcal{H}(0) = (0) = 1$ by limit. Moreover, $\mathcal{S}(t) \cdot \mathcal{H}(t) = \cosh(t)$. Another useful observation here is that

$$(2.6) \quad \mathcal{H}(t) \leq 1 + t, \quad t \geq 0$$

iii) Let $q \geq 1$ be a constant and f be a continuous function, we denote

$$(2.7) \quad \mathcal{I}_{K,N}(f, B_R, q) := r^2 \left(\int_{B_r} |f(x)|^{Nq} \nu(dx) \right)^{1/(Nq)}$$

Properties for this integral are given in §7.

• *Constants in Proofs.* The following constants shall be used frequently in the proof of Harnack inequalities (§8 – §10). They are *Fixed* for the entire paper. We also provide some rough estimates for them.

Remark 2.8. Notices The scenario making use the constants below is the following. Fix a large ball B_R , we shall perform some estimates on a small ball $B_r(x) \subset B_R$. All the constants shall depends on the \sqrt{KR} . But they does *Not* depend on size of the small ball. We *emphases* that all the constants below depend *Only* on \sqrt{KR} and N . They do *Not* depend on R *Nor* K in a separated way.

Denote

$$(2.8) \quad \alpha := N\mathcal{H}(\omega_{K,N}R) \leq 2R\sqrt{NK} + 1$$

Here we used Eq.(2.6).

$$(2.9) \quad \mu := \left[(18)^3 \alpha^2 (18)^\alpha \cosh(\omega R) \right]^{-N} \mathcal{D}_{4R}^{-4} \geq e^{-C\{R\sqrt{KN}+N^2\}}$$

Here, we have applied the estimate of α

$$(2.10) \quad M := 2\alpha^2(18)^\alpha \leq e^{C(R\sqrt{KN}+N)}$$

Again, we have employed the estimate of α .

$$(2.11) \quad \delta_0 := \left(2 \mathcal{D}_{2R}^{4/N} S(\omega r) \right)^{-1} \geq e^{-C(R\sqrt{KN}+1)}$$

$$(2.12) \quad C_3 := 2 \mathcal{D}_{2R} (M^{1/p_0} / \mu)^{1/N},$$

Note, since $\eta > 1$

$$(2.13) \quad \mathcal{D}_{2R} \frac{M}{\mu^{1/p_0}} \left(\frac{1}{C_3} \right)^{N\eta/p_0} < 1$$

$$(2.14) \quad p_1 := p_0 / (N\eta)$$

The last few constants are those appear in the statement of Thm.1.6 -Thm.1.8. We shall give explicit forms of $p_0, C_0, C_1(p_0), C_2$ here and in later proof, it shall be clear that these choices are sufficient. We shall also give some rough estimat of these constants.

Lemma 2.9. *Define $p_0, C_0, C_1(p_0), C_2$ as follows*

$$(2.15) \quad p_0 = \frac{1 - \log[1 + (e - 1)(1 - \mu)]}{\log M}, \quad C_0 := e^{2/p_0}$$

and

$$(2.16) \quad C_2 = C_1(p_0) := \left(3C_3 \sum_{k=0}^{\infty} \frac{1}{(1 + 1/M)^{kp_1}} \right)^{1/p_1} \frac{1}{\delta_0}$$

The following statement holds:

i) $1 + (M^{p_0} - 1) \sum_{k=0}^{\infty} (M^{p_0}(1 - \mu))^k = e$;

ii) $e^{1/p_0} \geq \delta_0 \geq 1$;

iii) *The constant p_0, C_0 satisfies*

$$p_0 \geq \frac{\mu}{4 \log M} \geq e^{-C[R\sqrt{KN}+N^2]}, \quad C_0 \leq \exp[e^{C[R\sqrt{KN}+N^2]}];$$

iv) *The constant $C_2 = C_1(p_0)$ satisfies*

$$C_2 = C_1(p_0) \leq \exp[e^{C(R\sqrt{KN}+N^2)}];$$

Proof. i) can be verified directly.

To show ii), note the relation

$$(2.17) \quad \frac{x}{e} \leq 1 - \log[e - x] \leq \frac{x}{e - 1}$$

we have then

$$\frac{1}{p_0} \geq \log M/\mu.$$

By the choice of M, μ, δ_0 , it is clear that

$$(2.18) \quad e^{1/p_0} \geq \frac{1}{\delta_0}.$$

To see iii), one just need to note $(1 + 1/M) \geq (1 + 1/M)^{p_1}$ for $p_1 < 1$.

By (Eq.2.17), it is then easy to show

$$p_0 \geq \frac{\mu}{4 \log M} \geq e^{-CN^2[R\sqrt{K/N}+1]}$$

The estimate of e^{1/p_0} then follows.

Finally to show v), consider the following manipulation. Denote $b := ((M + 1)/M)^{p_1}$

$$\begin{aligned} \sum_{k=0}^{\infty} \left(\frac{M}{M+1} \right)^{p_1 k} &= \frac{1}{1 - e^{-\log b}} = \frac{1}{2} + \frac{1}{2} \frac{1 + e^{-\log b}}{1 - e^{-\log b}} \\ &= \frac{1}{2} + \frac{1}{2} \coth\left[\frac{1}{2} \log(b)\right] = \frac{1}{2} + (\log b)^{-1} \mathcal{H}\left(\frac{1}{2} \log(b)\right) \\ &\leq 1 + \frac{1}{\log(b)} \leq \frac{3M}{p_1}. \end{aligned}$$

In the last two inequalities, we have used (Eq.2.6) and an estimat of $\log(1 + 1/M)$ similar to (Eq.2.17). Then, we can then easily estimate $C_2 = C_1(p_0)$ as stated. \square

3. CONTACT SET

In this section, we investigate properties of contact sets (recall Defn.1.1), in particular the behavior of the unknown function u on its associated contact sets. We shall see the contact sets recognize the underline metric geometry in an elegant way.

First, we state an alternative characterization of contact set, which has mentioned in [2].

Definition 3.1. The concave paraboloid $P_{a,y}$ of vertex y and opening a is a function of the form

$$P_{a,y} := -\frac{a}{2} \rho^2(x, y) + c_y, \quad c_y, a \in \mathbb{R}, \quad a \geq 0$$

Similarly, one defines convex paraboloid.

Proposition 3.2. *Let $u \in C(\overline{\Omega})$, $E \subset \Omega$ closed and $a \geq 0$. Then $x \in A(a, E, u)$ if and only if there exists a concave paraboloid $P_{a,y}$ of opening a and vertex $y \in E$ that touches u in $\overline{\Omega}$.*

Proof. Immediately follows from the definitions (Defn.1.1, Defn.2.7 and Defn.3.1). \square

The following proposition contains some basic properties of contact sets (see. [23]). Its proof is a routine check up and hence be omitted.

Lemma 3.3. *Let $u \in C(\overline{\Omega})$ and $E \subset \Omega$ closed, then*

a) For all $a \geq 0$, $A(a, E, u)$ is closed (hence ν -measurable).

b) If $u_k \rightarrow u$ uniformly in Ω , then

$$\limsup_{k \rightarrow \infty} A(a, E/\Omega, u_k) = \bigcap_{j=1}^{\infty} \bigcup_{k \geq j} A_k \subset A(a, E/\Omega, u).$$

c) if $a_k \rightarrow 0$, then

$$\limsup_{k \rightarrow \infty} A(a_k, E/\Omega, u) = \bigcap_{j=1}^{\infty} \bigcup_{k \geq j} A_k \subset A(0, E/\Omega, u).$$

d) if $E \subset F$, then

$$A(a, E/\Omega, u) \subset A(a, F/\Omega, u).$$

Remark 3.4. Though we shall only need a) of the previous lemma in this paper, the other three properties are of practical value. We shall illustrate some of their applications in a separated paper.

To consider the interaction between contact sets and the underline metric geometry, we shall need the following notion and proposition from standard Riemannian geometry.

We first recall the Hessian bound in support sense introduced by Calabi [5] (also see [20]).

Definition 3.5. Let $u \in C(\Omega)$. We say $D^2w \geq \beta g, \beta \in \mathbb{R}$ in support sense at x_0 if for every $\epsilon > 0$, there exists a smooth function φ_ϵ defined in a neighborhood of x_0 such that i) φ_ϵ touches w from below at x_0 . ii) $D^2\varphi_\epsilon(x_0) \geq (\beta - \epsilon)g(x_0)$. Similarly, one define $D^2w \leq \beta g$ in support sense.

The following well-known property of the distance function is useful (see. p.342 [20])

Proposition 3.6. *Let (\mathcal{M}, g) be a (smooth) Riemannian manifold. Given any $y \in \mathcal{M}$, $\nabla^2 \rho_y^2$ is locally bounded above in support sense, that is, for any compact set Z , there exists a constant \mathcal{L} (depending on $\text{diam}(Z)$ and the sectional curvature lower bound over Z), such that*

$$D^2 \rho_y^2(x) \leq \mathcal{L} g, \quad \forall x \in Z.$$

in support sense.

Remark 3.7. C^2 -functions are locally bounded above in support sense.

The following lemma contains a key feature of the contact sets.

Lemma 3.8. *Let $u \in C(\overline{\Omega})$. Let $E \subset \Omega$ be closed. Let $a \geq 0$. Suppose $A(a, E/\Omega, u) \subset \Omega$ and u is locally bounded above in support sense in Ω , then the following statement holds.*

if $y \in E$ and paraboloids $P_{a,y}$ touches u at $x \in A(a, E/\Omega, u)$, then x and y are neither cut-points nor focal points for each other and hence $P_{a,y}$ is smooth at x .

Proof. By the contact relation, the definition of bounded in support sense and Prop.3.6, there are two smooth function φ^+, φ_- such that φ^+ touches $P_{a,y}$ from above at x and φ_- touches it from below at x . It follows immediately that $P_{a,y}$ is differentiable at x . By the standard Riemannian geometry, x and y are not cut-points of each other.

To show x, y are not focal points to each other, consider the limit of the second order increment quotient

$$\Delta^2 P_{a,y}(x) := \limsup_{|W| \rightarrow 0} \frac{P_{a,y}(\exp_x W) + P_{a,y}(\exp_x -W) - 2P_{a,y}(x)}{|W|}.$$

The existence of φ^+, φ_- shows that

$$|\Delta^2 P_{a,y}(x)| < \infty$$

By Prop.2.5 of [18], this eliminates the possibility that x, y are focal points to each other. \square

Next lemma relates contact sets, the sub-level sets of u and the domain. Such a statement has indeed been used in [2].

Lemma 3.9. *Let $u \in C(\overline{B_r}(x_0))$. Let $u(y_0) = l$ for some $y_0 \in \overline{B_{r/2}}(x_0)$ and $u \geq t$ in $B_{5r/6}(x_0)$.*

Suppose $l < t$. then for any $a \geq 0$

$$A(a, \overline{B_{r/6}}(y_0)/B_r(x_0), u) \subset B_{5r/6}(x_0) \cap \{u \leq l + \frac{ar^2}{36}\}$$

Proof. Let P_{a,y_1} be a polynomial touching u at x_1 . By the contact relation, we have

$$(3.1) \quad u(y_0) \geq P_{a,y_1}(y_0) = -\frac{a}{2}\rho^2(y_0, y_1) + \frac{a}{2}\rho^2(x_1, y_1) + u(x_1).$$

Thus, we immediately have

$$u(x_1) \leq P_{a,y_1}(y_0) + \frac{a}{2}\rho^2(y_0, y_1) \leq l + \frac{ar^2}{36}.$$

So it suffices to show $x_1 \in B_{5r/6}(x_0)$.

Suppose on the contrary that $x_1 \in B_r(x_0) \setminus B_{5r/6}(x_0)$, then, by $y_0 \in B_{r/2}(x_0)$ and $y_1 \in B_{r/6}(y_0)$, we have

$$\rho(x_1, y_1) \geq \frac{r}{6},$$

Henceforth

$$(3.2) \quad \rho^2(x_1, y_1) - \rho^2(y_0, y_1) \geq 0$$

However, by (Eq.3.1)

$$(3.3) \quad \frac{a}{2}(\rho^2(x_1, y_1) - \rho^2(y_0, y_1)) \leq u(y_0) - u(x_1) \leq l - t < 0$$

Since $a \geq 0$, (Eq.3.2) contradicts to (Eq.3.3). \square

Remark 3.10. Though sufficing for this paper, this is not the most precise relation between contact sets and sub-level sets. However, the proof of above lemma has suggested how one might control the relative position of contact sets, sub-level sets and domain.

Remark 3.11. On space with special feature in metric geometry, such as Euclidean space which has parallelogram law, very precise relation can be draw regarding the relative location of contact sets with different opening (see. [23]).

4. JACOBI EQUATION AND JACOBI DETERMINANT

In this section, we quote some important results regarding the Jacobi equation and their geometrical implication from [25]. They are of fundamental importance in our development. In particular, we shall see how contact sets match with the Jacobi determinant (Prop.4.6). The content in this section follows closely to the Chapter 14 (p.365-372, p.379-383) in [25] and its third Appendix (p.412-418).

Definition 4.1. Let $R(t)$ be a t -dependent symmetric matrix. The Jacobi equation associate to $R(t)$ is the following ODE

$$(4.1) \quad \ddot{J}(t) + R(t) \cdot J(t) = 0.$$

Solutions to Eq.4.1 are called Jacobi matrices.

In the sequel, we shall always assume the time interval to be $[0, 1]$.

The following propositions contain the main properties of the Jacobi equation that supports our development (proof, see. p.429-432, [25]).

Proposition 4.2. Let J_0^1 and J_1^0 be Jacobi matrices defined by the initial conditions

$$J_0^1 = \dot{J}_1^0 = I, \quad \dot{J}_0^1 = J_1^0 = 0.$$

Assume J_1^0 is invertible for all $t \in (0, 1]$. Then,

$$S(t) := [J_1^0(t)]^{-1} J_0^1(t)$$

is symmetric for all $t \in (0, 1]$ and decreases monotonically.

Remark 4.3. The original statement in the book [25] also states that $S(t)$ is positive for all $t \in [0, 1]$. We have confirmed with the author that the statement $S(t)$ is positive is merely a typo. In particular, the material in the third Appendix of Ch.14 in [25] does not rely on positivity of $S(t)$.

Proposition 4.4. *Let $S(t)$ be the matrix defined in Prop.4.2. Let $J(t)$ be a Jacobi matrix satisfies the initial conditions*

$$J(0) = I, \quad \dot{J}(0) \text{ is symmetric}$$

Then the following properties are equivalent

- i) $\dot{J}(0) + S(1) \geq 0$;
- ii) $\det J(t) > 0$ for all $t \in [0, 1)$.

Now, we related the above pure ODE results to some geometry of Jacobi fields on Riemannian manifold. The following discussion follows closely to [25].

Let (\mathcal{M}, g) be a Riemannian manifold, given a geodesic $\gamma(t)$, one may parallel transport an orthonormal frame $e(0)$ at $T_{\gamma(0)}\mathcal{M}$ along $\gamma(t)$ to obtain frame $e(t)$ at $T_{\gamma(t)}\mathcal{M}$. Then, a family of sections $H(t) \in \text{Sym}T_{\gamma(t)}\mathcal{M}$ can be canonically identify to a family of symmetric matrices parametrized by t . In the rest of this paper, we shall always use this identification whenever necessary. Eigenvalues of $H(t)$ are independent of choice of the frame $e(t)$.

Consider the flow $F[u](t, \cdot)$:

Definition 4.5. Let $u \in C^2(\Omega)$. Define

$$F[u](t, \cdot) : \Omega \rightarrow \mathcal{M}, x \mapsto \exp_x(t\nabla u(x)).$$

For our convenience, we shall denote $F[u] = F[u](1, \cdot)$; and the notation $F_t[u]$ and $F[u](t, \cdot)$ are used interchangeably. We also denote the Jacobi transformation $dF[u]$ by $J[u]$.

The next proposition contains some geometric implication of the previous two propositions (see the discussion on p.413-414 in [25]).

Proposition 4.6. *Let $x, y \in \mathcal{M}$. Suppose x, y are neither cut-points nor focal points to each other. Then following statements holds:*

- i) $J[u](t, x)$ is a smooth (w.r.t time t) Jacobi matrix associated to

$$R_j^i(t, x) = \text{Riem}(\dot{\gamma}(t, x), e_i(t), \dot{\gamma}(t, x), e_j(t)).$$

with initial conditions

$$J(0, x) = I, \quad \dot{J}(0, x) = \nabla^2 u(x).$$

along the curve

$$\gamma[u](t, x) := \exp_x(t\nabla u(x))$$

- ii) if

$$\nabla^2 u(x) + \nabla^2 \left(\frac{1}{2} \rho_y^2 \right)(x) \geq 0$$

Then, $\det J[u](t, x) \geq 0$ for all $t \in [0, 1]$.

Proof. The proof is contained in Ch.14 of [25]. See the discussion on p.365– p.367 for i) and p.412– p.418 on ii) □

Remark 4.7. Note, this is a proposition regarding a Riemannian Manifold. No reference measure appear.

Remark 4.8. Jacobi fields has been used in [2] to give an explicit expression of $dF[u]$. Prop.4.6 can be deduced from that calculation. However, we shall not need the expression of $dF[u]$.

Next, we shall estimate the Jacobi determinant of $F[u]$ with respect to reference measure ν . Materials and their proofs can be found on Ch.14 of [25].

The Jacobi equation (4.1) immediately suggests that behavior of $J[u](t)$ is controlled by curvatures and the Hessian of u . However, we are only interested in estimating the Jacobi determinant, which can indeed be controlled by Ricci.

First, we make few definitions and notations.

Definition 4.9. Let (\mathcal{M}, g, ν) be a complete Riemannian Metric-Measure space and $\Omega \subset \mathcal{M}$ be a domain. Let $u \in C^2(\Omega)$. We denote

$$\mathcal{J}[u](t, x) := \det(J[u](t, x)), \quad x \in \Omega.$$

and define

$$\mathcal{J}_\nu[u](t, x) := \lim_{r \rightarrow 0} \frac{\nu[F_t[u](B_r(x))]}{\nu[B_r(x)]} = \frac{e^{-V(F_t[u](x))}}{e^{-V(x)}} \mathcal{J}[u](t, x), \quad x \in \Omega$$

Also, we denote

$$\mathcal{D}_N[u](t, x) = \begin{cases} (\mathcal{J}_\nu[u](t, x))^{1/N} & n \leq N < \infty \\ \log \mathcal{J}_\nu[u](t, x) & N = \infty \end{cases}.$$

The following propositions quoted from Ch.14 of [25] are the keys of estimating $\mathcal{J}_\nu[u]$,

Proposition 4.10. Let $u \in C^2(\Omega)$. Let $x \in \Omega$ and $\gamma(t, x)$ is a geodesic starting at x with $\dot{\gamma}(t, x) = \nabla u(x)$. Suppose $J[u](t, x)$ is invertible for all $t \in [0, 1)$, then

$$\ddot{\mathcal{D}}_n[u](t, x) \leq -\frac{\text{Ric}(\dot{\gamma}(t, x))}{n} \mathcal{D}_n[u](t, x), \quad \forall t \in (0, 1)$$

Proof. See page 368-370 of [25]. □

In the presence of reference measure, the techniques on page 380 [25] extends the above proposition as follows

Corollary 4.11. Let $u \in C^2(\Omega)$. Let $x \in \Omega$ and $\gamma(t, x)$ is a geodesic starting at x with $\dot{\gamma}(t, x) = \nabla u(x)$. Suppose $J[u](t, x)$ is invertible for all $t \in [0, 1)$, then, for any $N \in [n, \infty]$,

$$\ddot{\mathcal{D}}_N[u](t, x) \leq \begin{cases} -\frac{1}{N} \text{Ric}_{N, \nu}(\dot{\gamma}(t, x)) \mathcal{D}_N[u](t, x) & n \leq N < \infty \\ -\text{Ric}_{\infty, \nu}(\dot{\gamma}(t, x)) & N = \infty \end{cases}, \quad \forall t \in (0, 1)$$

Proof. See p.379-383 in [25]. □

Remark 4.12. Our definition of \mathcal{D}_∞ differs a sign from the function $l(t)$ given in [25].

5. MEASURE ESTIMATE

In this section, we give the proof of Thm.1.2. Indeed, all the technical work has been done in the previous two sections.

Proof of Thm.1.2. For convenience, write $A = A(a, E/B_r, u)$ in this proof. First, we show that the map $F[u](x) = F[u/a](1, x)$ is a subjective map from A onto E .

Fix a point $y \in E$, by definition, there exists a paraboloid $P_{a,y}$ touches u at some $x \in A$. By Lemma 3.8, we have $P_{a,y}$ is smooth at x and the contact condition implies

$$(5.1) \quad \nabla u(x) = -a\rho_y(x)\nabla\rho_y(x)$$

Hence,

$$F[u/a](1, x) = \exp_x[-\rho_y\nabla\rho_y(x)] = y.$$

This proves the subjectivity.

Next, as indicated by Lem.3.8 x and y are neither cut-points nor conjugate points of each other, Prop.4.6 along with the contact relation

$$\nabla^2 \frac{u}{a}(x) \geq -\nabla^2 \frac{1}{2}\rho_y^2(x)$$

imply that $J[u](t, x)$ is invertible for all $t \in (0, 1)$ and

$$(5.2) \quad \mathcal{J}_v(t, x) = \frac{e^{-V(F[u](t, x))}}{e^{-V(x)}} \det J[u](t, x) \geq 0 \quad \forall t \in [0, 1].$$

Therefore, \mathcal{D}_N (recall Defn.4.9) satisfies the differential inequality given in Cor.4.11.

Denote $\gamma(t, x) = \exp_x(t\nabla u(x)/a)$, by Eq.(5.1) and the fact γ is a geodesic, we have

$$|\dot{\gamma}(t, x)|^2 = |\dot{\gamma}(0, x)|^2 = \rho_y^2(x) \leq 4r^2, \quad \forall t \in [0, 1],$$

where the last inequality follows from the fact that $A \subset B_r$.

Therefore, along with the Ricci lower bound condition, the differential inequality in Cor.4.11 reduce to

$$(5.3) \quad \ddot{\mathcal{D}}_N[u/a](t, x) \leq \begin{cases} 4(K/N)r^2 & \mathcal{D}_N[\frac{u}{a}](t, x) \\ 4Kr^2 & N = \infty \end{cases}.$$

Now apply a standard ODE comparison argument with the initial condition that

$$\mathcal{D}_N[u/a](0, x) = \begin{cases} 1 & n \leq N < \infty \\ 0 & N = \infty \end{cases} \quad \dot{\mathcal{D}}_N[u/a](0, x) = \begin{cases} \frac{1}{Na}\Delta_v u(x) & n \leq N < \infty \\ \frac{1}{a}\Delta_v u(x) & N = \infty \end{cases},$$

we obtain

$$(5.4) \quad \mathcal{D}_N[u/a](1, x) \leq \mathcal{D}_{K,N,R}[u/a](x), \quad \forall x \in A, N \in [0, \infty]$$

Finally, we shall apply the Area formula. Since u is C^2 in Ω' containing A , $F[u/a](1, \cdot)$ is differentiable in Ω' . Thus, by the Area formula,

$$\nu[E] \leq \int_A \mathcal{J}_v(x) \nu(dx) \leq \int_A \left(\mathcal{D}_N[u/a](1, x) \right)^N \nu(dx) \quad n \leq N < \infty$$

and

$$\nu[E] \leq \int_A \mathcal{J}_\nu(x) \nu(dx) \leq \int_A \exp\left(\mathcal{D}_\infty[u/a](1, x)\right) \nu(dx), \quad N = \infty$$

Here, we have used (5.2) and the fact that $\mathcal{J}_\nu \geq 0$ (Eq.5.2). The desired formula follows from (Eq.5.4). \square

We provide some remarks on this estimate.

Remark 5.1. Note in the proof, the contact set is exactly where one can estimate Jacobi determinant of $F[u]$ —the place where $\det(dF[u])$ is nonnegative! This has already been observed by Cabré. In [2], one focus on working with sub-level sets, contact sets are used only as an intermediate step.

Remark 5.2. There are only two inequalities used in above proof. One is during the application of area formula. It reaches equality if and only if $\mathcal{F}[u]$ is one-to-one. Another one is the estimate by the ODE. This differential inequality (Eq.5.3) is, indeed, equivalent to the Ricci lower bounded (if it is satisfied by all suitable test functions, see details on p.400, Prop.14.8 in [25]).

Remark 5.3. The formula given by Thm.1.2 is, in certain sense, some dual formula to the Sobolev inequalities. In particular, the case $N = \infty$ could be viewed as a dual formula for the log-Sobolev inequality.

One way to recognize the duality is to consider the key ingredients in the proof of Thm.1.2 and the proof of Sobolev inequalities. It is known that Sobolev inequalities can be derived as consequence of Co-area formula. While Thm.1.2 is proved based on the Area formula.

Alternatively, one knows that Harnack inequalities are equivalent to Sobolev inequality on the manifolds satisfying doubling property (see, [22] and [11]), and we shall see in §8 – §10, Thm.1.2 along with the measure doubling property implies the Harnack inequality. Thus, there must be some relations between Thm.1.2 and Sobolev inequalities.

6. RICCI COMPARISON AND A BARRIER

In this section, we recall the Ricci comparison theorem for the modified Laplacian and use it to construct a barrier function which shall be used latter (proof of Lem.9.4).

Recall the following result in [21] (or [14], [27]).

Proposition 6.1. *Suppose $\text{Ric}_{N,\nu}|_{B_R} \geq -Kg$, $K \geq 0$, $N \in [0, \infty)$, then for any two points $x, y \in B_R$.*

$$\Delta_\nu \rho_y(x) \leq (N-1) \frac{\mathcal{H}(\omega_{K,N-1} \rho_y(x))}{\rho_y(x)}$$

in the support sense everywhere (recall notation 2.5 from §2).

Remark 6.2. Note, it is easy to check

$$1 + (N - 1)\mathcal{H}(\omega_{K,N-1}\rho) \leq N\mathcal{H}(\omega_{K,N}\rho)$$

Hence

$$\Delta_v \frac{\rho_y^2}{2}(x) \leq N\mathcal{H}(\omega_{K,N}\rho_y(x))$$

The rest of this section is devoted to construct a barrier function. A similar construction has been made in [2] (also adopted in [11]). However, as working with potentially negative curvature, one needs some more detailed information regarding such a barrier function to insure the constants depends on curvatures in a proper way (in particular, this is needed for considering elliptic fully-nonlinear PDEs). These information are obtained in the following lemmas. Their proofs are technical but completely routine.

Recall the definition of constant α from §2. Note $\alpha \geq 2$.

Lemma 6.3. *There exists a function $h : [0, \infty) \rightarrow \mathbb{R}$ such that*

i) $h \in C^2[0, \infty)$ and $h'(0) = 0$.

ii) $\inf_{[0, \infty)} h \geq -\alpha^2(18)^\alpha$

iii) The derivatives of h satisfies the following estimates:

For $t > 1/18$,

$$h''(t) - \frac{h'(t)}{t} = -\alpha(\alpha + 2)t^{-(\alpha+2)} < 0, \quad \frac{h'(t)}{t} = \alpha t^{-(\alpha+2)} > 0$$

For $0 \leq t \leq 1/18$,

$$|h''(t) - \frac{h'(t)}{t}| \leq 972 \alpha^2(18)^\alpha, \quad 0 < \frac{h'(t)}{t} \leq 972 \alpha^2(18)^\alpha,$$

Proof. Let $\beta_i, i = 0, 1, 2$ be constants to be determined. Consider the function

$$h(t) := \begin{cases} \beta_0 + \beta_1 t^2 + \beta_2 t^3 & t \leq \frac{1}{18} \\ (1/9)^{-\alpha} - t^{-\alpha} & t > \frac{1}{18} \end{cases},$$

By choosing

$$(6.1) \quad \beta_0 = -\frac{1}{6}\alpha(5 + \alpha)(18)^\alpha, \quad \beta_1 = \frac{(18)^2}{2}\alpha(3 + \alpha)(18)^\alpha, \quad \beta_2 = -\frac{(18)^3}{3}\alpha(2 + \alpha)(18)^\alpha$$

we match up the values and the first two derivatives of h at $t = 1/9$. It follows then h satisfies i).

Next, we estimate the derivatives of h . The case for $t > 1/9$ is clear. Consider $t \in [0, 1/9]$

Since $\alpha \geq 0$,

$$\beta_0, \beta_1 \geq 0 \quad \beta_2 < 0.$$

and,

$$-\frac{\beta_1}{\beta_2} = \frac{1}{12} \frac{(3 + \alpha)}{(2 + \alpha)} > \frac{1}{12}.$$

Therefore, we see,

$$\frac{h'(t)}{t} = 2\beta_1 + 3\beta_2 t$$

is strictly positive for $t \in [0, 1/18]$ and monotone decreasing; and

$$h''(t) - \frac{h'(t)}{t} = 3\beta_2 t$$

monotone decreasing and negative. The desired estimates follows from the expression of β_i 's and the fact $\alpha \geq 2$.

ii) follows from $h'(t) \geq 0$ in $0 \leq t \leq 1/18$ and the fact $\alpha \geq 2$. \square

Recall the definition of $\omega_{K,N}$ and α from §2. In the statement and the proof of next lemma, we shall denote $\omega_{K,N}$ by ω .

Lemma 6.4. *Let (\mathcal{M}, g, ν) be a complete Riemannian Metric-Measure space. Given geodesic ball $B_r(x_0)$ with $r \leq R$. Let $K \geq 0, 1 < N < \infty$. Suppose*

$$\text{Ric}_{N,\nu}|_{B_r(x_0)} \geq -Kg$$

Then there exists a function ψ such that:

i) ψ is continuous in $B_r(x_0)$ and lies in $C^2(B_r(x_0) \setminus \text{Cut}(x_0))$.

ii) $\inf_{B_r(x_0)} \psi \geq -\alpha^2(18)^\alpha$ and

$$\psi \geq (18)^\alpha - (4/3)^\alpha \text{ in } B_r(x_0) \setminus B_{3r/4}(x_0), \quad \psi = (18)^\alpha - 2^\alpha \text{ on } \partial B_{r/2}(x_0)$$

iii) ψ is locally bounded above in support sense in $B_r(x_0)$

iv) In $\overline{B}_{r/18}(x_0) \setminus \text{Cut}(x_0)$,

$$\frac{r^2 \Delta_\nu[\psi]}{N} + \mathcal{H}(\omega r) \leq 972\alpha^3 4^\alpha$$

v) In $B_r(x_0) \setminus (\overline{B}_{r/18}(x_0) \cup \text{Cut}(x_0))$

$$\frac{r^2 \Delta_\nu[\psi]}{N} + \mathcal{H}(\omega r) \leq 0$$

Proof. Let $h(t)$ be the function given in Lem.6.3 with α given in Eq.2.8. Denote ρ_{x_0} by ρ for convenience. Define

$$\psi := h(\rho/r).$$

We shall show ψ satisfies all desired properties.

Fix a poin $x \in B_r(x_0) \setminus \text{Cut}(x_0)$, all the expressions below are evaluated at this x . By direct calculation, we have

$$(6.2) \quad r^2 \nabla^2 \psi = \left(h'' - \frac{h'}{t} \right) \nabla \rho \otimes \nabla \rho + \frac{h'}{t} \nabla^2 \frac{\rho^2}{2}, \quad t = \rho/r$$

iii) follows from above equation and the standard argument of Calabi (p.282 [20]).

Follows immediately from Eq.6.2 and $\text{tr}(\nabla \rho \otimes \nabla \rho) = |\nabla \rho|^2 = 1$, we have

$$(6.3) \quad r^2 \Delta_\nu[\psi] = \left(h'' - \frac{h'}{t} \right) + \frac{h'}{t} \Delta_\nu \frac{\rho^2}{2}$$

If $x \in B_r(x_0) \setminus (B_{r/9}(x_0) \cup \text{Cut}(x_0))$, then by iii) of Lem.6.3, the inequality (6.3), and the Ricci comparison 6.1,

$$r^2 \Delta_v[\psi] \leq \alpha \left(\frac{\rho}{r} \right)^{-(\alpha+2)} \left(-\alpha - 2 + N\mathcal{H}(\omega r) \right)$$

Since $\alpha = N\mathcal{H}(\omega R) \geq N\mathcal{H}(\omega r) > 1$,

$$r^2 \Delta_v[\psi] \leq \alpha \left(-\alpha - 2 + N\mathcal{H}(\omega r) \right)$$

Thus

$$\frac{r^2 \Delta_v[\psi]}{N} + \mathcal{H}(\omega r) \leq \frac{\alpha}{N} (-\alpha - 2 + N\mathcal{H}(\omega r) + \frac{N\mathcal{H}(\omega r)}{\alpha})$$

v) follows from the very choice that $\alpha = N\mathcal{H}(\omega R) \geq N\mathcal{H}(\omega r)$ (Eq.2.8),

If $x \in B_{r/18}(x_0) \setminus \text{Cut}(x_0)$, then by the same calculation this time with the other part in iii) of Lem.6.3, we arrive (Note $N > 1$)

$$\frac{r^2 \Delta_v[\psi]}{N} + \mathcal{H}(\omega r) \leq \frac{972}{N} \alpha^2 9^\alpha + (972 \alpha^2 (18)^\alpha + 1) \mathcal{H}(\omega r)$$

iv) follows immediately from the choice of α . \square

7. MEASURE DOUBLING AND MONOTONICITY OF $\mathcal{I}_{K,N}$

In this section, we summarize some results regarding the measure doubling property, the integral $\mathcal{I}_{K,N}$ and some basic L^p theory that will be used in the proof of Harnack inequalities (Thm.1.6–Thm.1.8).

The following proposition estimates the doubling constant on a Riemannian Metric-Measure space in terms of Ricci lower bound (Cor.18.11, [25]).

Proposition 7.1. *Let (\mathcal{M}, g, ν) be a Riemannian metric-measure space, satisfying the curvature condition $\text{Ric}_{N,\nu} \geq -K$ for some $K \geq 0$ and $1 < N < \infty$. Denote \mathcal{D}_Ω the doubling constant in the domain Ω . Then ν is doubling with a constant \mathcal{D} :*

- $\mathcal{D}_\mathcal{M} \leq 2^N$ if $K = 0$;
- $\mathcal{D}_{B_R} \leq 2^N \left[\cosh \left(2 \sqrt{\frac{K}{N-1}} R \right) \right]^{N-1} \leq \mathcal{D}_{K,N,R}$ for any B_R if $K > 0$

(recall Eq.2.3 in §2).

The doubling property allows the following simple estimate. Recall $\eta = \eta_{K,N,R}$ and $\mathcal{D} = \mathcal{D}_{K,N,R}$ (Eq.2.3. Eq.2.4) from §2

$$(7.1) \quad \frac{\nu[B_{r_1}(x)]}{\nu[B_{r_2}(x)]} \leq \mathcal{D} \left(\frac{r_1}{r_2} \right)^{N\eta}$$

provide $B_{r_2}(x) \subset B_{r_1}(x) \subset B_R$ on a (\mathcal{M}, g, ν) with $\text{Ric}_{N,\nu}|_{B_R} \geq -Kg$.

Recall the definition of $\mathcal{I}_{K,N}$ (Eq.2.7) from §2. The integral $\mathcal{I}_{K,N}$ has good monotonicity and fits the scaling well.

Lemma 7.2. *For any $1 < N < \infty$, we have*

- i) $\mathcal{I}_N(f; B_r, t) \leq \mathcal{I}_N(f; B_r, s)$ whenever $t < s$.
- ii) If $B_{r_1}(x) \subset B_{r_2}(x) \subset B_R$, then

$$\mathcal{I}_N(f; B_{r_1}(x), \eta_{K,N,R}) \leq \mathcal{I}_N(f; B_{r_2}(x), \eta_{K,N,R}).$$

(recall the definition of $\eta_{K,N,R}$ (Eq.2.4) from §2).

Proof. i) follows from the standard L^p theory; ii) follows from direct calculation and Eq.7.1. \square

Given a function f on a domain Ω with finite-measure, denote

$$(7.2) \quad \tilde{\lambda}_\Omega(t) := \frac{\nu[\{f \leq t\} \cap \Omega]}{\nu[\Omega]}.$$

When no confusion arise, we shall omit the subscript.

We shall need the following well-known statement in L^p -theory (see [3] for instance).

Lemma 7.3. *Let $C > 1$. Then, for any $0 < p < \infty$,*

$$\int_\Omega f^p < \infty \Leftrightarrow S := \sum_{k=0}^{\infty} C^{pk} \tilde{\lambda}(C^k) < \infty$$

and

$$(1 - \frac{1}{C^p})S + \frac{1}{C^p} \tilde{\lambda}(1) \leq \int_\Omega f^p \leq 1 + (C^p - 1)S$$

8. PROOF OF HARNACK INEQUALITIES I

In this section, we establish the key lemma in proving Harnack inequalities (Thm.1.6–Thm.1.8). It describes the local growth of the unknown u . The similar lemma is used in [2]. Our proof is essentially same as that in [2] (also in [11]). However, by making use some fine properties of the contact sets, we avoid the approximation procedures needed in [2] and [11].

Recall from §2, the constant M, μ (Eq.2.9) and integral $\mathcal{I}_{K,N}$ (Eq.2.7).

Lemma 8.1. *Let (\mathcal{M}, g, ν) be a complete metric-measure space. Let $u \in C(\overline{B_{2R}}) \cap C^2(B_{2R})$ and $f \in C(B_{2R})$. Let $K \geq 0, N < \infty$.*

Suppose

$$\text{Ric}_{N,\nu}|_{B_{2R}} \geq -Kg, \quad \mathcal{I}_{K,N}(f, B_{2R}, 1) \leq \delta_0$$

Then, for any given ball $B_{2r}(x_0) \subset B_{2R}$,

$$u \geq 0 \text{ in } B_r(x_0), \quad \inf_{B_{r/2}} u \leq 1, \quad \Delta_\nu[u] \leq f \text{ in } B_r(x_0),$$

implies

$$(8.1) \quad \frac{\nu[\{u \leq M\} \cap B_{r/18}(x_0)]}{\nu[B_r(x_0)]} \geq \mu$$

Remark 8.2. Recall the **Notice** in §2, M, μ, δ_0 are constants defined w.r.t to the large ball B_{2R} . They are independent of the choice of the small ball $B_r(x)$.

Proof. Recall the definition of the constant α (§2 Eq.(2.8)). Let ψ be the function constructed in Lem.6.4 with respect to $B_r(x_0)$. Consider $w = u + \psi$ on $\subset B_r(x_0)$.

By the construction of ψ and the hypothesis of u , there exists $y_0 \in B_{r/2}(x_0)$ such that

$$w(y_0) = \inf_{B_{r/2}(x_0)} w \leq 1 + (18)^\alpha - 2^\alpha,$$

and w satisfies

$$\inf_{B_r(x_0) \setminus B_{3r/4}(x_0)} w \geq (18)^\alpha - \left(\frac{4}{3}\right)^\alpha.$$

This two condition along with Lem.3.9 and $\alpha \geq 2$ implies

$$(8.2) \quad A\left(\frac{1}{r^2}, \bar{B}_{r/6}(y_0)/B_r(x_0), w\right) \subset\subset B_r(x_0) \cap \{w \leq 1 + (18)^\alpha - 2^\alpha + \frac{1}{36}\}.$$

For convenience in the rest of the proof, we denote

$$A := A\left(\frac{1}{r^2}, \bar{B}_{r/6}(y_0)/B_r(x_0), w\right),$$

Recall ii) of Lem.6.4, we obtain

$$w = u + \psi \leq \frac{37}{36} + (18)^\alpha - 2^\alpha \Rightarrow u \leq 2\alpha^2(18)^\alpha$$

This along with (Eq.8.2) and the definition of M (§2, Eq.2.10) implies

$$(8.3) \quad \nu[A \cap B_{r/18}(x_0)] \leq \nu[\{u < M\} \cap B_{r/18}(x_0)].$$

Hence, it suffices to estimate $\nu[A \cap B_{r/18}(x_0)]$ from below.

Recall the definition of $\mathcal{D}_{K,N,r}, \omega_{K,N}$ (Eq.2.3) and the function $\mathcal{H}(t), \mathcal{S}(t)$ (Eq.2.5) from §2. In the rest of this proof, we shall denote $\omega_{K,N}$ by ω and $\mathcal{D}_{K,N,r}$ by \mathcal{D}_r . We will give the estimate for $\nu[A \cap B_{r/(18)}(x_0)]$ via Thm.1.2.

By Lem.6.4, ψ is locally bounded above in support sense. Also u is C^2 , hence by Prop.3.8, we have

$$A \cap \text{Cut}(x_0) = \emptyset$$

Since A is a closed (Lem.3.3) and $\text{Cut}(x_0)$ is also closed, there is a neighborhood $\Omega' \subset B_{r/4}(y_0)$ of A such that $w \in C^2(\Omega')$. Thus we may apply Thm.1.2 to obtain

$$(8.4) \quad \nu[B_{r/6}(y_0)] \leq \int_A \left\{ \mathcal{D}_{K,N,r}[r^2 w](x) \right\}^N \nu(dx)$$

with

$$\mathcal{D}_{K,N,r}[r^2 w](x) = \mathcal{S}(\omega r) \left[\mathcal{H}(\omega r) + \frac{r^2 \Delta_v w(x)}{N} \right]$$

Since $\Delta_v u \leq f$, we have

$$\mathcal{H}(\omega r) + \frac{r^2 \Delta_v w(x)}{N} \leq \frac{r^2 f(x)}{N} + \mathcal{H}(\omega r) + \frac{r^2}{N} \Delta_v \psi(x).$$

Lem.6.4 implies: for any $x \in A \cap \overline{B}_{r/18}(x_0)$

$$(8.5) \quad \mathcal{D}_{K,N,r}[r^2 w](x) \leq 972\alpha^3 4^\alpha \mathcal{S}(\omega r) + \mathcal{S}(\omega r) \left(\frac{r^2 f^+(x)}{N} \right);$$

and for any $x \in A \cap (B_r(x_0) \setminus \overline{B}_{r/18}(x_0))$,

$$(8.6) \quad \mathcal{D}_{K,N,r}[r^2 w](x) \leq \mathcal{S}(\omega r) \left(\frac{r^2 f^+(x)}{N} \right).$$

These estimates along with the simple relation

$$(t + s)^N = (t + s)^{+,N} \leq 2^{N-1}(t^{+,N} + s^{+,N}), \quad t + s > 0,$$

implies

$$\begin{aligned} \int_A \left\{ \mathcal{D}_{K,N,r}[r^2 w](x) \right\}^N \nu(dx) &\leq \frac{1}{2} \left[(18)^3 \alpha^2 (18)^\alpha \cosh(\omega r) \right]^N \nu[A \cap B_{r/18}(x_0)] \\ &\quad + 2^{N-1} \mathcal{S}^N(\omega r) \int_{B_r(x_0)} \left(\frac{r^2 f^+(x)}{N} \right)^N \end{aligned}$$

Combine with (Eq.8.4), we obtain

$$(8.7) \quad \begin{aligned} 1 &\leq \frac{1}{2} \left[(18)^3 \alpha^2 4^\alpha \cosh(\omega r) \right]^N \frac{\nu[A \cap B_{r/18}(x_0)]}{\nu[B_{r/6}(y_0)]} \\ &\quad + 2^{N-1} \mathcal{S}^N(\omega r) \frac{1}{\nu[B_{r/6}(y_0)]} \int_{B_r(x_0)} \left(\frac{r^2 f^+(x)}{N} \right)^N \end{aligned}$$

By the doubling property

$$(8.8) \quad \nu[B_r(x_0)] \leq \nu[B_{3r/2}(y_0)] \leq \mathcal{D}_{2r}^4 \nu[B_{r/6}(x_0)]$$

we see, by monotonicity of $\mathcal{I}_{K,N}$ (Lem.7.2),

$$(8.9) \quad \begin{aligned} \frac{1}{\nu[B_{r/6}(y_0)]} \int_{B_r(x_0)} \left(\frac{r^2 f^+(x)}{N} \right)^N &\leq \mathcal{D}_{2r}^4 \int_{B_r(x_0)} \left(\frac{r^2 f^+(x)}{N} \right)^N \\ &\leq \mathcal{D}_{2r}^4 \left(\mathcal{I}_{K,N}(f, B_r, 1) \right)^N \leq \mathcal{D}_{2r}^4 \delta_0^N \end{aligned}$$

Therefore, by combine (Eq.8.7) with (Eq.7) and recall the choice of δ_0 (Eq.2.11, in §2), we arrive

$$\frac{1}{2} \left[(18)^3 \alpha^2 (18)^\alpha \cosh(2\omega r) \right]^N \mathcal{D}_{2r}^4 \frac{\nu[A \cap B_{r/18}(x_0)]}{\nu[B_r(x_0)]} \geq 1/2$$

Recalling the definition of μ (§2, Eq.2.9) and the inequality (8.8), we have then

$$(8.10) \quad \frac{\nu[A \cap B_{r/18}(x_0)]}{\nu[B_r(x_0)]} \geq \left[(18)^3 \alpha^2 (18)^\alpha \cosh(\omega r) \right]^{-N} \mathcal{D}_{2r}^{-4} \geq \mu$$

The proof is completed by jointing the above inequality with (Eq.8.3). \square

9. PROOF OF HARNACK INEQUALITIES II

In this section, we shall prove Thm.1.6. The idea here is essentially same as that in [2], which dates back to [12] and [13]. Our presentation here follows [23].

First, we recall the following version of the Vitali's covering lemma. One may refer to standard textbook in measure theory for a proof.

Lemma 9.1. *Let (X, ρ, ν) be a metric-measure space; Let \mathcal{V} be a family of closed balls of nonzero radius in X and D be the collection of centers of these balls.*

Suppose

$$\sup\{\text{diam}(B) : B \in \mathcal{V}\} < \infty,$$

and ν satisfies the local measure doubling property, that is, for any compact set Z , ν has a doubling constant depends on Z .

Then there exists a countable subcollection \mathcal{V}' of \mathcal{V} such that

$$D \subset \bigcup_{B \in \mathcal{V}'} B$$

and the collection

$$\{\frac{1}{4}B, B \in \mathcal{V}'\}$$

is disjoint.

Remark 9.2. The local measure doubling property is only used to insure the collection \mathcal{V}' is countable.

Recall the definition of constant M , μ and integral $\mathcal{I}_{K,N}(f)$ from §2. The Thm.1.6 follows immediately from the following two lemmas

Lemma 9.3. *Under assumption of Thm.1.6. Denote*

$$D_k := \{x \in B_{R/2} : u(x) \leq M^k\}.$$

Suppose additionally that

$$\inf_{B_{R/2}} u \leq 1 \text{ and } \mathcal{I}_{K,N}(f, B_{2R}, \eta) \leq \delta_0.$$

Then for any $k \geq 0$

$$\nu[D_{k+1} \cap B_{r_x/4}(x)] \geq \mu\nu[B_{r_x}(x)]$$

for all $x \in B_{R/2} \setminus D_k$ and $r_x = \text{dist}(x, D_k)$.

Proof. Fix $x_0 \in B_{R/2} \setminus D_k$. In the rest of the proof, write

$$r_0 = r_{x_0} = \text{dist}(x_0, D_k)$$

for convenience. Since $D_0 \neq \emptyset$, we have

$$(9.1) \quad r_0 \leq R/2$$

Denote z_0 for the center of $B_{R/4}$. Connecting x_0 and z_0 by a minimizing geodesic. Choose y_0 be a point on this geodesic such that

$$\rho(y_0, x_0) = r_0/8;$$

and consider the ball

$$B_{r_0/8}(y_0).$$

By triangle inequality and the estimate of r_0 and the fact (minimizing geodesic)

$$\rho(y_0, z_0) + \rho(y_0, x_0) = \rho(z_0, x_0),$$

we see

$$B_{r_0/8}(y_0) \subset B_{r_0/4}(x_0) \cap B_{R/2}.$$

Therefore

$$(9.2) \quad B_{r_0/4}(x_0) \cap D_{k+1} \supset B_{r_0/8}(y_0) \cap \{w \leq M\}.$$

where

$$w = u/M^k.$$

Thus, it suffices to estimate

$$\nu[B_{r_0/8}(y_0) \cap \{w \leq M\}]$$

from below.

Consider the ball

$$B_l(y_0), \quad l = \frac{9}{4}r_0$$

Firstly, as $r_0 < R/2$, we have $l \leq \frac{9}{8}R$ and by triangle inequality,

$$B_{2l}(y_0) \subset B_{2R}.$$

Secondly, as

$$\text{dist}(y_0, D_k) \leq \rho(x_0, y_0) + \text{dist}(x_0, D_k) \leq \frac{9}{8}r_0.$$

we have, as $l/2 = 9r_0/8$,

$$\overline{B}_{l/2} \cap D_k \neq \emptyset.$$

Thirdly, note

$$\frac{l}{18} = \frac{r_0}{8}.$$

hence

$$B_{l/18}(y_0) \cap \{w \leq M\} = B_{r_0/8}(y_0) \cap \{w \leq M\}$$

With these three elementary relations, we may apply the Lem.8.1 to w on $B_l(y_0)$ to obtain

$$(9.3) \quad \nu[B_{r_0/8} \cap \{w \leq M\}] \geq \mu \nu[B_l(y_0)]$$

Finally, use the triangle inequality again, we see

$$(9.4) \quad B_l(y_0) = B_{9r_0/4}(y_0) \supset B_{r_0}(x_0).$$

Combine Eq.9.2, Eq.9.3 and Eq.9.4, we complete the proof. \square

Lemma 9.4. *Under assumption of Thm.1.6. Suppose additionally that*

$$\inf_{B_{R/2}} u \leq 1 \text{ and } \mathcal{I}_{K,N}(f; B_{2R}; \eta) \leq \delta_0.$$

Then for any $k \geq 0$

$$\tilde{\lambda}_{B_{R/2}}(M^k) \leq (1 - \mu)^k.$$

Proof. Recall

$$D_k := \{x \in B_{R/2} : u(x) \leq M^k\}.$$

Claim: for any $k \geq 0$

$$\nu[(D_{k+1} \setminus D_k) \cap B_{R/2}] \geq \mu \nu[B_{R/2} \setminus D_k].$$

Consider the cover \mathcal{V} of the set $B_{R/2} \setminus D_k$ defined by

$$\mathcal{V} := \{\bar{B}_{r_x}(x) : x \in B_{R/2} \setminus D_k, r_x := d(x, D_k)\}.$$

By Lem.9.3, we have

$$\nu[D_{k+1} \cap B_{r_x/4}(x)] \geq \mu \nu[B_{r_x}(x)]$$

By Vitali's covering lemma, we may take a sequence of ball $B_{r_i} \in \mathcal{V}$ such that $B_{r_i/4}(x_i)$ are disjoint to each other and

$$B_{R/2} \setminus D_k \subset \cup_i B_{r_i}(x_i).$$

Moreover, by the choice of r_x , we also have $B_{r_i/4}(x_i) \cap D_k = \emptyset$ and henceforth,

$$\left(\bigcup_i B_{r_i/4} \cap B_{R/2} \right) \subset B_{R/2} \setminus D_k$$

Now, we compute

$$\begin{aligned} \nu[B_{R/2} \setminus D_k] &\leq \nu[\cup_i B_{r_i}(x_i)] \leq \sum_i \nu[B_{r_i}(x_i)] \leq \sum_i \frac{1}{\mu} \nu[D_{k+1} \cap B_{r_i/4}(x_i)] \\ &= \frac{1}{\mu} \nu[\cup_i (D_{k+1} \cap B_{r_i/4}(x_i))] = \frac{1}{\mu} \nu[D_{k+1} \cap (\cup_i B_{r_i/4}(x_i))] \\ &\leq \frac{1}{\mu} \nu[D_{k+1} \cap (B_{R/2} \setminus D_k)]. \end{aligned}$$

Here, at the first equality in the second line, we have used the fact that $B_{r_i/4}$ are disjoint. Now we have proved the claim.

Recall the definition of $\tilde{\lambda}(t) = \tilde{\lambda}_{B_{R/2}}(t)$ (Eq.7.2), we immediately obtain from the claim that

$$\tilde{\lambda}(M_R^{k+1}) \leq (1 - \mu) \tilde{\lambda}(M_R^k).$$

The desired estimate follows by inductively apply this inequality. \square

Proof of Thm.1.6. Denote $\eta = \eta_{K,N,2R}$. Replacing u by

$$\tilde{u} = \frac{u}{e^{1/p_0}(\inf_{B_{R/2}} u + \mathcal{I}_{K,N}(f, B_{2R}, \eta))},$$

we have

$$\Delta \tilde{u} \leq [e^{1/p_0}(\inf_{B_{R/2}} u + \mathcal{I}_{K,N}(f; B_{2R}; \eta))]^{-1} f = \tilde{f}.$$

Then, recall ii) of Lem.2.9, \tilde{u} satisfies

$$\inf_{B_{R/2}} u \leq 1 \quad \mathcal{I}_{K,N}(\tilde{f}, B_{2R}, \eta) \leq \frac{1}{e^{1/p_0}} \leq \delta_0$$

Hence, by this normalization, may assume

$$\inf_{B_{R/2}} u \leq 1 \quad \mathcal{I}_{K,N}(f; B_{2R}; \eta) \leq \delta_0.$$

Therefore, it suffices to bound averaged- L^{p_0} -norm of u by e^{1/p_0} .

Recall i) of Lem.2.9, p_0 satisfies

$$1 + (M^{p_0} - 1) \sum_{k=0}^{\infty} M^{p_0 k} \tilde{\lambda}(M^k) \leq e$$

Applying Lem.9.4, Lem.7.3 along with the above identity, we have

$$\left(\int_{B_{R/2}} u^{p_0} \right)^{1/p_0} \leq e^{1/p_0}.$$

The desired estimate is then obtained by re-normalizing u back and recall the choice of constant C_0 . \square

10. PROOF OF HARNACK INEQUALITIES III

In this section, we give the proof of Thm.1.7 and Theorem 1.8.

Recall the constant p_0 in the statement of Thm.1.6 (see Lem.2.9). Recall the constant δ_0, M, μ, p_1 from §2. Again, they are chosen w.r.t to the large ball B_{2R} . We denote $\eta_{K,N,2R}$ (Eq.2.4) by η in the rest of this section.

The key part is the following lemma.

Lemma 10.1. *Under the assumption of Thm.1.7. Suppose additionally*

$$\left(\int_{B_R} u^{p_0} \right)^{1/p_0} \leq 1, \quad \mathcal{I}_{K,N}(f, B_{2R}, \eta) \leq \delta_0.$$

and

$$\beta := u(x_0) > M, \quad B_{r_0}(x_0) \subset B_R \text{ with } r_0 = RC_3 \beta^{-p_1}$$

Then,

$$\sup_{B_{r_0}(x_0)} u \geq \beta(1 + 1/M).$$

Proof. Argue by contradiction. Suppose

$$\sup_{B_{r_0}(x_0)} u < \beta(1 + 1/M)$$

Consider the function

$$w = \frac{\beta(1 + 1/M) - u}{\beta/M}.$$

on $B_{r'}(y_0)$ with

$$r' = r_0/4, \quad \rho(y_0, x_0) = r_0/8.$$

Note that w satisfies

$$\Delta_\nu w \leq \frac{-f}{\beta/M} \leq |f| \text{ in } B_{r'}(x_0)$$

and

$$w \geq 0 \text{ on } B_{r'}(y_0), \quad \inf_{B_{r'/2}(y_0)} w \leq w(x_0) \leq 1$$

Thus, we may apply the Lem.8.1 to obtain

$$(10.1) \quad \mu \nu[B_{r'}(y_0)] \leq \nu[\{w \leq M\} \cap B_{r'/18}(y_0)]$$

Observe that

$$w \leq M \Rightarrow u \geq \beta/M.$$

Hence, (Eq.10.1) along with the Chebyshev's inequality implies

$$\beta \leq \frac{M}{\mu^{1/p_0}} \left(\int_{B_{r'}(y_0)} u^{p_0} \right)^{1/p_0} \leq \frac{M}{\mu^{1/p_0}} \left(\frac{\nu[B_R]}{\nu[B_{r'}(y_0)]} \right)^{1/p_0}.$$

Now, recall the choice of $r' = r_0/4$, p_1 (Eq.2.14 in §2) and apply the doubling estimate (Eq.7.1) to $B_{r'}(y_0) \subset B_R(y_0) \subset B_{2R}$, we obtain

$$\beta \leq \frac{M}{\mu} \mathcal{D}_{2R} \left(\frac{R}{r'} \right)^{N\eta/p_0} \leq \frac{M}{\mu^{1/p_0}} \mathcal{D}_{2R} \left(\frac{1}{C_3} \right)^{N\eta/p_0} \beta$$

Recalling the choice of C_3 (Eq.2.12, Eq.2.13), then the above inequality implies $\beta < \beta$, which is impossible. \square

Both Thm.1.7 and Thm.1.8 can be deduced from Lem.10.1. Recall the constants M, p_1, C_3, μ from §2.

Proof of Thm.1.7. We first prove Thm.1.7 with $p = p_0$. By replacing u by

$$\tilde{u} = \left\{ \left(\int_{B_R} u^{p_0} \right)^{1/p_0} + \mathcal{I}_{K,N}(f; B_{2R})/\delta_0 \right\}^{-1} u,$$

Similar to the proof of Thm.1.6, we may assume u satisfies the hypothesis in Lem.10.1. Thus, it suffices to bound $\sup u$ by the constant $C_1(p_0)$ (Eq.2.16) given in §2.

Argue by contradiction. Suppose there exists $x_0 \in B_{R/2}$ such that

$$(10.2) \quad \beta := u(x_0) > C_1(p_0).$$

Consider the sequence x_k defined as follows: choose inductively x_{k+1} such that

$$u(x_{k+1}) = \sup_{B_{r_k}(x_k)} u, \quad r_k = RC_3 u(x_k)^{-p_1}$$

Consider the sum,

$$(10.3) \quad RC_3 \sum_{k=0}^{\infty} \frac{1}{\beta^{p_1} (1 + 1/M)^{kp_1}}$$

The hypothesis on β (Eq.10.2) and the choice of $C_1(p_0)$ together implies

$$RC_3 \sum_{k=0}^{\infty} \frac{1}{\beta^{p_1} (1 + 1/M)^{kp_1}} \leq R/3.$$

Hence, we may inductively apply the Lem.10.1 to obtain

$$u(x_k) \geq C_1(p_0)(1 + 1/M)^k, \quad r_k < \frac{RC_3}{C_1^{p_1}(p_0)(1 + 1/M)^{p_1 k}}.$$

and

$$\sum_k r_k \leq \frac{R}{3}.$$

However, this implies

$$x_k \in B_{r_k}(x_k) \subset B_R, \quad \forall k \geq 0$$

and $u(x_k)$ tend to ∞ in B_R . This contradicts the fact that u is continuous in $\overline{B_R}$ and hence prove the case $p = p_0$.

For $p > p_0$, one may take $C_1(p) = C_1(p_0)$ and the desired inequality follows from standard L^p theory. For $p < p_0$, one may apply a standard interpolation argument (eg. Ch.4 of [10]). In that argument, one shall need the factor $\eta = \eta_{K,N,2R}$ given by the doubling property and get

$$C_1(p) = \tilde{C}_1(p, N, \eta) C_1(p_0).$$

Note η only depends on \sqrt{KR} . □

Proof of Thm.1.8. By replacing u by

$$\frac{u}{C_0(\inf_{B_{R/2}} u + \mathcal{I}_{K,N}(f; B_R))},$$

Similar to the proof of Thm.1.6, we may assume

$$\inf_{B_{R/2}} u + \mathcal{I}_{K,N}(f; B_R) \leq \frac{1}{C_0}, \quad \mathcal{I}_{K,N}(f; B_R) \leq \delta_0$$

By Thm.1.6, we obtain

$$\left(\int_{B_{R/2}} u^{1/p_0} \right)^{1/p_0} \leq 1$$

Using Lem.10.1 and following the exactly same argument as in the proof of Thm.1.7, we obtain

$$\sup_{B_{R/2}} u \leq C_1(p_0) = C_2$$

The desired estimate is achieved by re-scaling u back. \square

11. CHARACTERIZATION OF RICCI LOWER BOUND

In this section, we formulate some questions regarding characterizing Ricci lower bound by the Harnack inequality.

We first define the Harnack functional which forms the foundation of our discussion.

Definition 11.1. Let (\mathcal{M}, g) be a Riemannian manifold. The Harnack functional associate to the metric g is defined by

$$(11.1) \quad \mathcal{H}_g(r, x) := \sup_u \left\{ \frac{\sup_{B_{r/2}^g(x)} u}{\inf_{B_{r/2}^g(x)} u} : \Delta_g u = 0, u > 0 \text{ in } B_r^g(x) \right\}$$

where $r > 0$ and $x \in \mathcal{M}$.

Remark 11.2. The superscript and superscript g is to emphases that B^g is the geodesic ball w.r.t to the metric g and Δ_g is the Laplacian w.r.t to g .

Remark 11.3. From the Harnack inequality (Thm.1.8), we see \mathcal{H} is well-defined. It is the best Harnack constant *for the given Riemannian manifold*. Note here, the Riemannian manifold is fixed in priori. There are several Harnack inequalities with sharp constant exists in the lecture. However, the sharp there means for a given K , there is a manifold and a harmonic function to realize the inequality. So they cannot be used to calculate \mathcal{H} in explicit.

Denote $\lambda_1[\text{Ric}_g](x)$ the smallest eigenvalue of Ric_g at x . Our idea is to consider asymptotic behaviors of $\mathcal{H}_g(r, x)$ for r small. We believe \mathcal{H}_g 's asymptotic behavior should characterize $\lambda_1[\text{Ric}_g]$. More precisely, we believe that fix given point x , expansion of $\mathcal{H}_g(r, x)$ near $r = 0$ characterizing the smallest eigenvalue of Ric at the point. The concrete questions are follows.

Question 1: Given a smooth Riemannian Manifold (\mathcal{M}, g) . Fix a $x \in \mathcal{M}$, does $\mathcal{H}_g(r, x)$ have a series expansion near $r = 0$? Suppose this is the case,

$$\mathcal{H}_g(r, x) = a_0 + a_1 r + a_2 r^2 + O(r^3), \quad r \rightarrow 0$$

Then, whether the following statement holds:

- i) Is a_1 always zero for all (\mathcal{M}, g) and for any $x \in \mathcal{M}$;
- ii) Is there a universal constant \mathbf{h} only depends on dimension, does Not vary according to (\mathcal{M}, g) nor point x , such that

$$\mathbf{h} a_2(x, g) = \lambda_1[\text{Ric}_g](x).$$

- iii) Can \mathcal{H}_g be extended as a even function w.r.t to r ?

The above questions might be too surprising from some aspects. However, we do believe they have affirmative answers, at least for i) and ii). The following proposition gives some simple evidence of our guess.

Proposition 11.4. *i) On \mathbb{R}^2 with standard Euclidean metric g_0 , $\mathcal{H}_{g_0}(r, x)$ is constant in both argument and*

$$\mathcal{H}_{g_0} = 9$$

ii) On \mathbb{S}^2 the sphere with metric g_K whose Ricci is constant $K \geq 0$ (see. Eq. complex Ricci), $\mathcal{H}_{g_K}(r, x)$ is constant in x and has an even extension w.r.t r . Moreover, it has the following expansion

$$\mathcal{H}_{g_K}(r, x) = 9 - 3Kd^2 + \frac{3}{8}K^2d^4 - \frac{11}{480}K^3d^6 + O(d^7), \quad d \rightarrow 0$$

iii) On \mathbb{H}^2 the Hyperbolic plane with metric g_{-K} whose Ricci is constant $-K$ with $K \geq 0$, $\mathcal{H}_{g_{-K}}(r, x)$ is constant in x and has an even extension w.r.t r . Moreover, it has the following expansion

$$\mathcal{H}_{g_{-K}}(r, x) = 9 - 3(-K)d^2 + \frac{3}{8}(-K)^2d^4 - \frac{11}{480}(-K)^3d^6 + O(d^7), \quad d \rightarrow 0$$

Proof. In \mathbb{R}^2 , Harmonic functions are invariant under the scaling: $v(x) := u(rx)$ and the translation $v(x) = u(x + y_0)$. Hence clearly from the definition. $\mathcal{H}(r, x)$ is a constant independent of x and r . It has a trivial expansion. and its $a_2 = 0$ is the curvature of \mathbb{R}^2 .

By the Poisson integral formula

$$u(x) := \frac{1 - |x|^2}{2\pi} \int_0^{2\pi} \frac{u(\omega)d\omega}{|x - e^{i\omega}|^2}$$

one may give a *sharp* estimate that

$$(11.2) \quad \frac{\sup_{B_{\theta r}(x)} u}{\inf_{B_{\theta r}(x)} u} \leq \frac{(1 + \theta)^2}{(1 - \theta)^2}$$

In particular take $\theta = 1/2$. we proves i).

To consider \mathbb{S}^2 and \mathbb{H}^2 , it is convenient to consider the conformal formulation and use complex coordinates.

Recall \mathbb{S}^2 can be represented as

$$(\mathbb{C}, g_K = \sigma^2(z)dz \wedge d\bar{z}), \quad \sigma_K(z) := \frac{\sqrt{2}}{(1 + K|z|^2)}$$

whose Ricci curvature is

$$(11.3) \quad \text{Ric} = -\partial\bar{\partial} \log \sigma^2(z) = \frac{2K}{(1 + K|z|^2)^2} = K\sigma^2$$

By the homogeneity of \mathbb{S}^2 , it suffices to consider $\mathcal{H}_{g_K}(r, 0)$.

Recall the relation between the geodesic $B_d^{g_K}(0)$ and Euclidean ball $B_r(0)$ (in this proof balls without superscript means Euclidean ball)

$$B_d^{g_K}(0) = B_r(0), \text{ where } r = \tan(\sqrt{K}d/\sqrt{2})$$

This relation follows easily from integrate $\sigma(t)$ along rays in \mathbb{R}^2 and observes that rays in \mathbb{R}^2 passing 0 is geodesic on \mathbb{S}^2 .

Also recall the Laplacian operator Δ_{g_K} , w.r.t to the conform metric, is of the following form

$$\Delta_{g_K} = \frac{1}{\sigma^2} \Delta (\text{the standard Laplacian}).$$

Combine these two relations, we see positive harmonic functions on $B_d^{g_K}(0)$ are positive harmonic functions on $B_r(0)$ with $r = \tan(d/\sqrt{2})$ and vice versa. Hence,

$$\mathcal{H}_{g_K}(d, 0) = \sup_u \left\{ \frac{\sup_{B_{r'}(x)} u}{\inf_{B_{r'}(x)} u} : \Delta u = 0, u > 0 \text{ in } B_r(x) \right\}$$

where $r' = \tan(d/(2\sqrt{2}))$ and $r = \tan(d/\sqrt{2})$. Therefore apply the sharp Harnack estimate with

$$\theta := \frac{r'}{r} = \frac{\tan(\sqrt{K}d/(2\sqrt{2}))}{\tan(\sqrt{K}d/\sqrt{2})}$$

We obtain

$$\mathcal{H}_{g_K}(d, 0) = \left[1 + 2 \cos\left(\frac{\sqrt{K}d}{\sqrt{2}}\right) \right]^2$$

which is clearly even; By direct calculation, we obtain the desired expansion.

The argument for Hyperbolic 2-pane is exactly same. This time

$$\mathbb{H} = (\mathbf{D} = \{|z| < 1\}), \quad g_{-K} = \frac{2}{(1 - K|z|^2)^2} dz \wedge d\bar{z}.$$

The correspondence between geodesic distance d from 0 and Euclidean distance r is

$$\sqrt{2K}d = \log \frac{1+r}{1-r}.$$

With the exactly same way of calculation, we obtain

$$\mathcal{H}_{g_{-K}}(d, 0) = \left[1 + 2 \cosh\left(\frac{\sqrt{K}d}{\sqrt{2}}\right) \right]^2$$

iii) then follows. □

Remark 11.5. The above calculation method is also work for the metric $g = (1+|z|^2)^2 dz \wedge d\bar{z}$ and gives desired expansion of $\mathcal{H}_g(r)$ at $z = 0$. Note this metric is not constant curvature any more.

Certainly, the above examples are very special; and the method to check that their Harnack functional have desired properties is very limited. However, they give some positive evidence for our guess.

Intuitively, we think the first order coefficient a_1 should always vanishes. Indeed, letting r tending to zero is equivalent to consider an infinitesimal ball. But the metric geometry of a Riemannian manifold does not differ from Euclidean space in first order approximation. Thus, the Harnack inequality for this infinitesimal ball should agree with that on Euclidean space up to first order.

It is clear that all the formulations in this section have their corresponding part for BE-Ricci curvature. However, in the presence of non-trivial reference measure, one also need to take the effective dimension into count. We believe the Harnack functional defined similar to Defn.11.1 shall relate to the product \sqrt{KN} . It should related to the optimal effective dimension of the Riemannian metric measure space

It seems also of interests to consider the asymptotic behavior of $\mathcal{H}_g(r)$ for r tending to ∞ on non-compact spaces. This asymptotic behavior of \mathcal{H}_g should indicate some information on the way $\lambda_1[\text{Ric}_g]$ distributed on the manifold. However, this issue is quite subtle and we are currently of no idea about it.

We believe the study of Harnack functional will have applications in the study of geometric flow, Gromov-Hausdroff convergence and Alexandrov spaces.

12. APPENDIX: FULLY-NONLINEAR UNIFORM ELLIPTIC EQUATIONS

In this appendix, we briefly explain how the proof in this paper might be extended to cover fully-nonlinear uniform elliptic equations on Riemannian manifolds ($\nu = \text{vol}_g$).

Recall from the standard theory of fully-nonlinear PDEs, it suffices to prove the Harnack inequalities for the Pucci-extremal operator

Definition 12.1. Let $\theta \geq 1$ be a constant and $u \in C^2$. The Pucci extremal operator, denoted by \mathcal{M}_θ^+ , is defined by

$$\begin{aligned}\mathcal{M}_\theta^-[u](x) &:= \mathcal{M}_\theta^-(H) := \sum_{\lambda_i \geq 0} \lambda_i(H) + \theta \sum_{\lambda_i \leq 0} \lambda_i(H) \\ \mathcal{M}_\theta^+[u](x) &:= \mathcal{M}_\theta^+(H) := \sum_{\lambda_i < 0} \lambda_i(H) + \theta \sum_{\lambda_i \geq 0} \lambda_i(H)\end{aligned}$$

where $H = \nabla^2 u(x)$ and $\lambda_i(H)$ denote eigenvalues of H .

The next lemma demonstrate the generality and the extremity of the Pucci operator (see [3]).

Lemma 12.2. Let $\text{Sym } T\mathcal{M}$ be the bundle of g -self-adjoint operators on $T\mathcal{M}$. Let $H \in \text{Sym } T\mathcal{M}$ be a section

$$\begin{aligned}\mathcal{M}_\theta^-(H) &:= \inf\{\text{tr}[A \cdot H], A \in \text{Sym } T\mathcal{M}, Id \leq A \leq \theta Id\} \\ \mathcal{M}_\theta^+(H) &:= \sup\{\text{tr}[A \cdot H], A \in \text{Sym } T\mathcal{M}, Id \leq A \leq \theta Id\}\end{aligned}$$

For the purpose of investigating Pucci operator, the following quantity is very convenient (see. [11])

$$(12.1) \quad \mathcal{E}_\theta(r) := \sup_{(x,y)} \{ \mathcal{M}_\theta^+ [\nabla^2(\frac{1}{2}\rho_y^2)](x) - \text{tr}[\nabla^2(\frac{1}{2}\rho_y^2)](x), \rho(x,y) \leq r \}$$

The following lemma explains some property of this quantity

Lemma 12.3. *Let (M, g) be a complete Riemannian manifold. Then the following statement is true*

i) *If $\nabla^2(\frac{1}{2}\rho_y^2)(x)$ is non-negative definite for all x, y with $x, y \in B_R$, then*

$$\mathcal{E}_\theta(2R) \leq (\theta - 1)(1 + (n - 1)\mathcal{H}(\omega_{K,n}R))$$

where K is the Ricci lower bounded in B_R . In particular, this holds for any ball B_R with $R \leq R_s$, where R_s depends on sectional curvature upper bounded in B_{2R} .

ii) *Suppose in B_R , sectional curvatures are bounded below by $-K_s$ with $K_s \geq 0$. Then*

$$\mathcal{E}_\theta(2R) \leq (\theta - 1)(1 + (n - 1)\mathcal{H}(\sqrt{K_s/n}R)).$$

Proof. The first part of i) follows immediately from the definition of $\mathcal{E}_\theta(R)$, Ricci comparison. The second part of i) follows by applying sectional curvature comparison against spheres.

ii) follows from standard sectional curvature comparison. \square

The proof of Thm.1.6 – Thm.1.8 can be easily extended to prove the following Harnack inequalities for Pucci extremal operator.

Theorem 12.4. *Let (\mathcal{M}, g, ν) be a Complete Smooth Riemannian Metric-Measure Space. Let $K \geq 0$ be a constant. Let $u \in C^2(B_{2R}) \cap C(\overline{B_R})$ and $f \in C(B_{2R})$. Suppose*

$$\text{Ric}|_{B_{2R}} \geq -K$$

Then, by replacing \sqrt{K} in the dependence of p_0, C_1, C_2 (not including η) with

$$R\sqrt{K} + \mathcal{E}_\theta(2R)$$

the following statements holds:

i) $\mathcal{M}_\theta^-[u] \leq f$ and $u \geq 0$ in B_{2R} implies the inequality (Eq.1.3);

ii) $\mathcal{M}_\theta^+[u] \geq f$ in B_{2R} implies the inequality (Eq.1.4);

iii) $\mathcal{M}_\theta^-[u] \leq |f|$ and $\mathcal{M}_\theta^+[u] \geq f$ and $u \geq 0$ in B_{2R} implies the inequality (Eq.1.5);

Remark 12.5. In case $\text{Sec}|_{B_{2R}} = 0$, Thm.12.4 along with the Lem.12.3 recovers the Harnack inequality proved in [2]. Thm.12.4 clearly fulfill the claim made in [2] regarding the space with sectional curvature lower bound. Indeed, Thm.12.4 can be viewed as an extension of [11].

Remark 12.6. Thm.12.4 and Lem.12.3 together implies that when the domain B_{2R} is small enough (depend only on sectional curvature lower bound), then the estimate only requires local Ricci lower bound.

Remark 12.7. Unlike the divergence case, in case $\theta \neq 1$, we think the dependence on $\mathcal{E}_\theta(r)$ (hence sectional curvature) can NOT be replaced by Ricci curvature. This can be seen in the following way. Fix some $x \in \mathcal{M}$, if Hessian $H(x) := \nabla^2(\frac{1}{2}\rho_y^2(x))$ has a negative eigenvalue, then by choosing θ large,

$$\mathcal{M}_\theta^+(H) \sim \theta \lambda_n(H), \mathcal{M}_\theta^-(H) \sim \theta \lambda_1(H)$$

where λ_1, λ_n is the smallest and largest eigenvalue of H . Thus, $\mathcal{M}_\theta^+, \mathcal{M}_\theta^-$ are affected by the extremal eigenvalues of H . While in divergence case, the operator ($Id \leq A \leq \theta A$)

$$\int g(A \nabla \frac{\rho_y^2}{2}, \nabla \varphi) \sim \theta \rho_y |\nabla \varphi|, \quad \varphi \text{ test function}$$

is only affected by ρ . From this comparison, the dependence on $\mathcal{E}_\theta(R)$ seems necessary.

Proof of Thm 12.4. Recall the proof in §9 – §10 only relies on the Lem.8.1, hence it suffices to prove Lem.8.1 for \mathcal{M}_θ^- with $R \sqrt{K}$ in the constant μ, M replaced by

$$R \sqrt{K} + \mathcal{E}_\theta(2R).$$

Recall the proof of Lem.8.1, we then see it suffices to control Δw from above on contact set $A = A(a, B_{r/6}(y_0)/B_r(x_0), w)$ (recall $w = u + \psi$) by $\mathcal{M}_\theta^- u \leq f$ and $R \sqrt{K} + \mathcal{E}_\theta$. This can be done by the following simple calculation

At $x \in A$, by contact condition, we have

$$\nabla^2 w(x) \geq -a \nabla^2(\frac{1}{2}\rho_y^2)(x),$$

Denote $S = \nabla_y^2 w(x)$ and $H = H_y^v(x)$. We then have $S + aH \geq 0$.

The result follows from the following calculation.

$$\begin{aligned} \Delta u(x) &= \text{tr}[S + aH - aH] = \text{tr}[S + aH] - a \text{tr}[H] \\ &\leq \mathcal{M}_\theta^-(S + aH) - a \text{tr}[H] \leq \mathcal{M}_\theta^-(S) + a \mathcal{E}_\theta(2r) \end{aligned}$$

While $S = \nabla^2 u + \nabla^2 \psi$, by the elementary inequality regarding Pucci operator, we have

$$\mathcal{M}_\theta^-(S) \leq \mathcal{M}_\theta^-[u](x) + \mathcal{M}^+[\psi](x)$$

Following the construction of the barrier ψ (Lem.6.4), one can easily see

$$r^2 \mathcal{M}^+[\psi] \leq -\alpha(\alpha + 2) + a \mathcal{E}_\theta(2r).$$

Hence by replacing $\sqrt{K}R$ with $\sqrt{K}R + \mathcal{E}_\theta(R)$ in the expression of α , Δu is controlled on the contact set in the desired way.

The rest of the proof of the Thm.12.4 follows line by line from our previous proof §9 – §10. \square

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